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Research Article

QUANTUM HERMITE-HADAMARD TYPE INEQUALITY AND SOME ESTIMATES OF QUANTUM MIDPOINT TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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ABSTRACT

In this paper, we give the correct quantum Hermite-Hadamard type inequality for the functions of two variables over finite rectangles. We provide some quantum estimates between the middle and the leftmost terms in correct quantum Hermite-Hadamard inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard type inequality on co-ordinates, Quantum Hermite-Hadamard type inequality, Quantum Hermite-Hadamard type inequality on co-ordinates, Convexity on co-ordinates, Quasi-convexity on co-ordinates.

1. INTRODUCTION

Quantum calculus (named q -calculus) is the study of calculus without limits. In [5], Jackson started study of q -calculus and presented q -definite integrals. The topic of q -calculus has wide applications in different areas of mathematics and physics. Some recent developments in the theory of q -calculus and theory of inequalities in q -calculus see [3, 4, 6]. The most recently, many authors started developing quantum integral inequalities using classical convexity and quasi-convexity (see [1, 10, 11, 12, 13, 15, 16, 17, 18]).

In, [10], Latif et al. develop quantum integral inequalities theory for functions of two variables and provide some q -Hermite-Hadamard type inequality of functions of two variables over finite rectangles. Also, Latif et al. provide some quantum estimates for the rightmost terms of

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the q -Hermite-Hadamard type inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.

Throughout this paper, for the conciseness we will suppose that $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, $0 < q < 1$, $0 < q_1 < 1$ and $0 < q_2 < 1$ are constants, $\Delta := [a, b] \times [c, d] \subset \mathbb{R}^2$ is a rectangle and $\Delta^\circ := (a, b) \times (c, d)$ is the interior of Δ .

Let real function f be defined on some non-empty interval I of real line \mathbb{R} . The function f said to be convex on I , if the inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

The function f said to be quasi-convex on I , if the inequality

$$f(ta + (1-t)b) \leq \sup\{f(a), f(b)\}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

The most important integral inequality for convex functions is the Hermite-Hadamard inequality, which is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

A function $f: \Delta \rightarrow \mathbb{R}$ is called convex (quasi-convex) on the co-ordinates if the partial mappings $f_y: [a, b] \rightarrow \mathbb{R}$, $f_y(u) := f(u, y)$ and $f_x: [c, d] \rightarrow \mathbb{R}$, $f_x(v) := f(x, v)$ convex (quasi-convex) where defined for all $y \in [c, d]$ and $x \in [a, b]$ (see [2, 14]).

A formal definition for the co-ordinated convex functions is given in [7, Definition 1] as follows:

Definition 1. [7] A function $f: \Delta \rightarrow \mathbb{R}$ is said to be co-ordinated convex on Δ , for all $t, s \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequality holds:

$$\leq tsf(x, y) + s(1-t)f(x, v) + t(1-s)f(u, y) + (1-t)(1-s)f(u, v).$$

Similarly, a formal definition for the co-ordinated quasi-convex functions is given in [9, Definition 2] as follows:

Definition 2. [9] A function $f: \Delta \rightarrow \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ , for all $t, s \in [0, 1]$ and $(x, y), (u, v) \in \Delta$, if the following inequality holds:

$$f(ty + (1-t)x, sv + (1-s)u) \leq \sup\{f(x, y), f(x, v), f(u, y), f(u, v)\}.$$

In [2], Dragomir proves the Hermite-Hadamard type inequality for co-ordinated convex functions as follows:

Theorem 1. Suppose that $f: \Delta \rightarrow \mathbb{R}$ is the co-ordinated convex on Δ . Then one has the following inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \leq \\ &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ &\left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \leq \\ &\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \quad (1.2)$$

The above inequalities are sharp.

In [8], Latif and Dragomir prove some new inequalities which give estimate between the middle and the leftmost terms in (1.2) for differentiable the co-ordinated convex functions, by using the following identity.

Lemma 1. Let $f: \Delta \rightarrow \mathbb{R}$ be a partial differentiable mapping. If $\frac{\partial^2 f}{\partial s \partial t} \in L(\Delta)$, then the following identity holds:

$$\mu(a, b, c, d) = (b-a)(d-c) \int_0^1 \int_0^1 K(t, s) \frac{\partial^2 f(ta+(1-t)b, sc+(1-s)d)}{\partial s \partial t} ds dt \quad (1.3)$$

where

$$\mu(a, b, c, d) = \left[\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & - \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \end{aligned} \right],$$

and

$$K(t, s) = \begin{cases} ts & , \quad (t, s) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \\ t(s-1) & , \quad (t, s) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right] \\ s(t-1) & , \quad (t, s) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \\ (t-1)(s-1) & , \quad (t, s) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right] \end{cases}.$$

In [1], Alp et al. prove the correct q -Hermite-Hadamard inequality as follows:

Theorem 2. Let $f: [a, b] \rightarrow \mathbb{R}$ be a convex differentiable function on (a, b) and $0 < q < 1$. Then we have

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}. \quad (1.4)$$

In the same paper, Alp et al. obtain inequalities for q -differentiable convex and quasi-convex mappings which are connected with the left hand part of the inequality (1.4).

2. PRELIMINARIES AND DEFINITIONS OF q -CALCULUS IN THE PLANE

The following definitions and properties for partial $q_1, q_2, q_1 q_2$ -derivatives and $q_1 q_2$ -integral of a function f on Δ are given in [10].

Definition 3. [10] Let $f: \Delta \rightarrow \mathbb{R}$ be a continuous function of two variables, the partial q_1 -derivatives, q_2 -derivatives and $q_1 q_2$ -derivatives of f at $(x, y) \in \Delta$ can be defined as follows:

$$\frac{{}_a \partial_{q_1} f(x, y)}{{}_a \partial_{q_1} x} = \frac{f(q_1 x + (1-q_1)a, y) - f(x, y)}{(1-q_1)(x-a)}, \quad x \neq a, \quad (2.1)$$

$$\frac{{}_c \partial_{q_2} f(x, y)}{{}_c \partial_{q_2} y} = \frac{f(x, q_2 y + (1-q_2)c) - f(x, y)}{(1-q_2)(y-c)}, \quad y \neq c, \quad (2.2)$$

$$\frac{{}_a {}_c \partial_{q_1 q_2} f(x, y)}{{}_a \partial_{q_1} x {}_c \partial_{q_2} y} = \frac{\begin{bmatrix} f(q_1 x + (1-q_1)a, q_2 y + (1-q_2)c) \\ -f(q_1 x + (1-q_1)a, y) \\ -f(x, q_2 y + (1-q_2)c) + f(x, y) \end{bmatrix}}{(1-q_1)(1-q_2)(x-a)(y-c)}, \quad x \neq a, y \neq c. \quad (2.3)$$

The function $f: \Delta \rightarrow \mathbb{R}$ is said to be partially $q_1, q_2, q_1 q_2$ -differentiable on Δ° if $\frac{{}_a \partial_{q_1} f(x, y)}{{}_a \partial_{q_1} x}$, $\frac{{}_c \partial_{q_2} f(x, y)}{{}_c \partial_{q_2} y}$ and $\frac{{}_a {}_c \partial_{q_1 q_2}^2 f(x, y)}{{}_a \partial_{q_1} x {}_c \partial_{q_2} y}$ exist for all $(x, y) \in \Delta^\circ$ respectively.

Definition 4. [10] Let $f: \Delta \rightarrow \mathbb{R}$ be a continuous function of two variables. Then the definite $q_1 q_2$ -integral on Δ is defined by

$$\int_c^y \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s = (1 - q_1)(1 - q_2)(x - a)(y - c) \quad (2.4)$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m f(q_1^n x + (1 - q_1^n)a, q_2^m y + (1 - q_2^m)c)$$

for $(x, y) \in \Delta$. If $(x_1, y_1) \in \Delta^\circ$, then

$$\int_{y_1}^y \int_{x_1}^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s = \int_{y_1}^y \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s - \int_{y_1}^y \int_a^{x_1} f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s =$$

$$\int_c^y \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s - \int_c^{y_1} \int_a^x f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s - \int_c^y \int_a^{x_1} f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s +$$

$$\int_c^{y_1} \int_a^{x_1} f(t, s) {}_a d_{q_1} t {}_c d_{q_2} s. \quad (2.5)$$

In [10], Latif et al. prove the quantum Hermite-Hadamard inequality of functions of two variables over finite rectangles as follows:

Theorem 3. Suppose that $f: \Delta \rightarrow \mathbb{R}$ is the co-ordinated convex on Δ . Then one has the following inequalities

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2} y \quad (2.6)$$

$$\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x$$

$$\leq \left[\frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a, y) {}_c d_{q_2} y + \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(b, y) {}_c d_{q_2} y \right]$$

$$\left[+ \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, c) {}_a d_{q_1} x + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, d) {}_a d_{q_1} x \right]$$

$$\leq \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1+q_1)(1+q_2)}.$$

The first and second inequalities in (2.6) are not correct. In this paper, our aim is to give correct quantum Hermite-Hadamard type inequality of functions of two variables over finite rectangles and provide some quantum estimates between the middle and the leftmost terms in correct quantum Hermite-Hadamard inequalities of functions of two variables using convexity and quasi-convexity on the co-ordinates.

3. NEW QUANTUM HERMITE-HADAMARD TYPE INEQUALITIES ON THE CO-ORDINATES

Throughout this section, we will take

$$\mu_{q_1, q_2}(a, b, c, d)(f) := f\left(\frac{q_1 a + b}{1 + q_1}, \frac{q_2 c + d}{1 + q_2}\right) - \frac{1}{d - c} \int_c^d f\left(\frac{q_1 a + b}{1 + q_1}, y\right) {}_c d_{q_2} y$$

$$- \frac{1}{b - a} \int_a^b f\left(x, \frac{q_2 c + d}{1 + q_2}\right) {}_a d_{q_1} x + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x.$$

Theorem 4. (Quantum Hermite-Hadamard inequalities on the co-ordinates) Let $f: \Delta \rightarrow \mathbb{R}$ is the co-ordinated convex and partially differentiable function on Δ . Then we have the following inequalities

$$f\left(\frac{q_1 a + b}{1 + q_1}, \frac{q_2 c + d}{1 + q_2}\right) \quad (3.1)$$

$$\leq \frac{1}{2(b-a)} \int_a^b f\left(x, \frac{q_2 c + d}{1 + q_2}\right) {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f\left(\frac{q_1 a + b}{1 + q_1}, y\right) {}_c d_{q_2} y$$

$$\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \leq \left[\frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a, y) {}_c d_{q_2} y + \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(b, y) {}_c d_{q_2} y \right. \\ \left. + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, c) {}_a d_{q_1} x + \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, d) {}_a d_{q_1} x \right] \leq \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1+q_1)(1+q_2)}.$$

The above inequalities are sharp.

Proof. Since f is the co-ordinated convex function on Δ and partially differentiable function on Δ° , it follows that the function $f_x: [c, d] \rightarrow \mathbb{R}$, $f_x(y) := f(x, y)$ is convex on $[c, d]$ and differentiable function on (c, d) for all $x \in [a, b]$. Then by using the q -Hermite-Hadamard inequality (1.4), one has

$$f_x \left(\frac{q_2 c + d}{1+q_2} \right) \leq \frac{1}{d-c} \int_c^d f_x(y) {}_c d_{q_2} y \leq \frac{q_2 f_x(c) + f_x(d)}{1+q_2}, \quad x \in [a, b]. \quad (3.2)$$

That is

$$f \left(x, \frac{q_2 c + d}{1+q_2} \right) \leq \frac{1}{d-c} \int_c^d f(x, y) {}_c d_{q_2} y \leq \frac{q_2 f(x, c) + f(x, d)}{1+q_2}, \quad x \in [a, b]. \quad (3.3)$$

By q_1 -integrating of (3.3) on $[a, b]$, we have

$$\frac{1}{b-a} \int_a^b f \left(x, \frac{q_2 c + d}{1+q_2} \right) {}_a d_{q_1} x \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \leq \frac{1}{1+q_2} \left[\frac{q_2}{b-a} \int_a^b f(x, c) {}_a d_{q_1} x + \frac{1}{b-a} \int_a^b f(x, d) {}_a d_{q_1} x \right]. \quad (3.4)$$

By a similar argument applied for the function $f_y: [a, b] \rightarrow \mathbb{R}$, $f_y(x) := f(x, y)$, we get

$$\frac{1}{d-c} \int_c^d f \left(\frac{q_1 a + b}{1+q_1}, y \right) {}_c d_{q_2} y \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \leq \frac{1}{1+q_1} \left[\frac{q_1}{d-c} \int_c^d f(a, y) {}_c d_{q_2} y + \int_c^d f(b, y) {}_c d_{q_2} y \right]. \quad (3.5)$$

Summing the inequalities (3.4) and (3.5), we have the following inequalities

$$\frac{1}{2(b-a)} \int_a^b f \left(x, \frac{q_2 c + d}{1+q_2} \right) {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f \left(\frac{q_1 a + b}{1+q_1}, y \right) {}_c d_{q_2} y \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x \leq \frac{q_2}{2(1+q_2)(b-a)} \int_a^b f(x, c) {}_a d_{q_1} x + \frac{1}{2(1+q_2)(b-a)} \int_a^b f(x, d) {}_a d_{q_1} x + \frac{q_1}{2(1+q_1)(d-c)} \int_c^d f(a, y) {}_c d_{q_2} y + \frac{1}{2(1+q_1)(d-c)} \int_c^d f(b, y) {}_c d_{q_2} y. \quad (3.6)$$

Also by using the q -Hermite-Hadamard inequality (1.4), we have

$$f \left(\frac{q_1 a + b}{1+q_1}, \frac{q_2 c + d}{1+q_2} \right) \leq \frac{1}{b-a} \int_a^b f \left(x, \frac{q_2 c + d}{1+q_2} \right) {}_a d_{q_1} x \quad (3.7)$$

and

$$f \left(\frac{q_1 a + b}{1+q_1}, \frac{q_2 c + d}{1+q_2} \right) \leq \frac{1}{d-c} \int_c^d f \left(\frac{q_1 a + b}{1+q_1}, y \right) {}_c d_{q_2} y. \quad (3.8)$$

Summing the inequalities (3.7) and (3.8), we have the following inequality

$$f \left(\frac{q_1 a + b}{1+q_1}, \frac{q_2 c + d}{1+q_2} \right) \leq \frac{1}{2(b-a)} \int_a^b f \left(x, \frac{q_2 c + d}{1+q_2} \right) {}_a d_{q_1} x + \frac{1}{2(d-c)} \int_c^d f \left(\frac{q_1 a + b}{1+q_1}, y \right) {}_c d_{q_2} y. \quad (3.9)$$

Finally by using the q -Hermite-Hadamard inequality (1.4), we have

$$\frac{q_2}{2(1+q_2)} \left(\frac{1}{b-a} \int_a^b f(x, c) {}_a d_{q_1} x \right) \leq \frac{q_2}{2(1+q_2)} \left(\frac{q_1 f(a, c) + f(b, c)}{1+q_1} \right), \quad (3.10)$$

$$\frac{q_2}{2(1+q_2)} \left(\frac{1}{b-a} \int_a^b f(x, d) {}_a d_{q_1} x \right) \leq \frac{q_2}{2(1+q_2)} \left(\frac{q_1 f(a, d) + f(b, d)}{1+q_1} \right), \quad (3.11)$$

$$\frac{q_1}{2(1+q_1)} \left(\frac{1}{d-c} \int_c^d f(a, y) {}_c d_{q_2} y \right) \leq \frac{q_1}{2(1+q_1)} \left(\frac{q_2 f(a, c) + f(a, d)}{1+q_2} \right), \quad (3.12)$$

$$\frac{q_1}{2(1+q_1)} \left(\frac{1}{d-c} \int_c^d f(b, y) {}_c d_{q_2} y \right) \leq \frac{q_1}{2(1+q_1)} \left(\frac{q_2 f(b, c) + f(b, d)}{1+q_2} \right). \quad (3.13)$$

Summing the inequalities (3.10)-(3.13), we have the following inequality

$$\begin{aligned} & \frac{q_2}{2(1+q_2)} \left(\frac{1}{b-a} \int_a^b f(x, c) {}_a d_{q_1} x \right) + \frac{q_2}{2(1+q_2)} \left(\frac{1}{b-a} \int_a^b f(x, d) {}_a d_{q_1} x \right) \\ & + \frac{q_1}{2(1+q_1)} \left(\frac{1}{d-c} \int_c^d f(a, y) {}_c d_{q_2} y \right) + \frac{q_1}{2(1+q_1)} \left(\frac{1}{d-c} \int_c^d f(b, y) {}_c d_{q_2} y \right) \\ & \leq \frac{q_1 q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1+q_1)(1+q_2)}. \end{aligned} \quad (3.14)$$

By combining (3.6), (3.9) and (3.14), we have (3.1). Thus the proof is accomplished. ■

Remark 1. In the Theorem 4, if one takes limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then one has the Theorem 1.

Lemma 2. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}_a c \partial_{q_1, q_2}^2 f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s}$ is continuous and integrable on Δ , then the following equality holds:

$$\begin{aligned} & \mu_{q_1, q_2}(a, b, c, d)(f) = q_1 q_2 (b-a)(d-c) \\ & \times \left[\int_0^1 \int_0^1 \kappa(t, s) \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right] \end{aligned} \quad (3.15)$$

where

$$\kappa(t, s) = \begin{cases} ts & , (t, s) \in \left[0, \frac{1}{1+q_1}\right] \times \left[0, \frac{1}{1+q_2}\right] \\ t \left(s - \frac{1}{q_2}\right) & , (t, s) \in \left[0, \frac{1}{1+q_1}\right] \times \left(\frac{1}{1+q_2}, 1\right] \\ s \left(t - \frac{1}{q_1}\right) & , (t, s) \in \left(\frac{1}{1+q_1}, 1\right] \times \left[0, \frac{1}{1+q_2}\right] \\ \left(t - \frac{1}{q_1}\right) \left(s - \frac{1}{q_2}\right) & , (t, s) \in \left(\frac{1}{1+q_1}, 1\right] \times \left(\frac{1}{1+q_2}, 1\right] \end{cases}$$

Proof. It is clear that

$$\begin{aligned} & q_1 q_2 (b-a)(d-c) \left[\int_0^1 \int_0^1 \kappa(t, s) \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right] \\ & = q_1 q_2 (b-a)(d-c) \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} ts \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right. \\ & + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(t - \frac{1}{q_1}\right) \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\ & + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(s - \frac{1}{q_2}\right) \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\ & \left. + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(t - \frac{1}{q_1}\right) \left(s - \frac{1}{q_2}\right) \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right] \\ & = q_1 q_2 (b-a)(d-c) \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} ts \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right. \\ & + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} ts \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\ & + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 ts \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\ & \left. + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 ts \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right] \end{aligned} \quad (3.16)$$

$$\begin{aligned}
 & + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 t s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - \frac{1}{q_1} \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - \frac{1}{q_2} \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - \frac{1}{q_2} \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 t \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - \frac{1}{q_1} \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & + \frac{1}{q_1 q_2} \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \Big] \\
 & = \left[q_1 q_2 (b-a)(d-c) \int_0^1 \int_0^1 t s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right. \\
 & - q_2 (b-a)(d-c) \int_0^1 \int_0^1 s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - q_1 (b-a)(d-c) \int_0^1 \int_0^1 t \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & + q_2 (b-a)(d-c) \int_0^1 \int_0^{\frac{1}{1+q_1}} s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & + q_1 (b-a)(d-c) \int_0^1 \int_0^{\frac{1}{1+q_1}} t \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & + (b-a)(d-c) \int_0^1 \int_0^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - (b-a)(d-c) \int_0^{\frac{1}{1+q_1}} \int_0^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & - (b-a)(d-c) \int_0^1 \int_0^{\frac{1}{1+q_1}} \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & \left. + (b-a)(d-c) \int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \right].
 \end{aligned}$$

We use Definition 3 and Definition 4 to calculate the appearing last nine integrals in (3.16). In terms of brevity, we will omit the details.

$$q_1 q_2 (b-a)(d-c) \int_0^1 \int_0^1 t s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \quad (3.17)$$

$$\begin{aligned}
 & = \\
 & -f(b, d) - \frac{1}{b-a} \int_a^b f(x, d) {}_0 d_{q_1} x - \frac{1}{d-c} \int_c^d f(b, y) {}_0 d_{q_2} y + \\
 & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) {}_c d_{q_2} y {}_a d_{q_1} x,
 \end{aligned}$$

$$\begin{aligned}
 & q_2 (b-a)(d-c) \int_0^1 \int_0^1 s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & = -f(b, d) - \frac{1}{d-c} \int_c^d f(b, y) {}_0 d_{q_2} y,
 \end{aligned} \quad (3.18)$$

$$\begin{aligned}
 & q_1 (b-a)(d-c) \int_0^1 \int_0^1 t \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & = -f(b, d) - \frac{1}{b-a} \int_a^b f(x, d) {}_0 d_{q_1} x,
 \end{aligned} \quad (3.19)$$

$$q_2 (b-a)(d-c) \int_0^{\frac{1}{1+q_1}} \int_0^1 s \frac{{}_a c \partial_{q_1, q_2}^2 f(t b + (1-t) a, s d + (1-s) c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \quad (3.20)$$

$$\begin{aligned}
 &= -f\left(\frac{q_1 a + b}{1 + q_1}, d\right) - \frac{1}{d - c} \int_c^d f\left(\frac{q_1 a + b}{1 + q_1}, y\right) {}_0 d_{q_2} y, \\
 & q_1(b - a)(d - c) \int_0^1 \int_0^{\frac{1}{1 + q_1}} t \frac{{}_a c \partial_{q_1, q_2}^2 f(tb + (1 - t)a, sd + (1 - s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 & (3.21) \\
 &= -f\left(b, \frac{q_2 c + d}{1 + q_2}\right) - \frac{1}{b - a} \int_a^b f\left(x, \frac{q_2 c + d}{1 + q_2}\right) {}_0 d_{q_1} x, \\
 & (b - a)(d - c) \int_0^1 \int_0^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(tb + (1 - t)a, sd + (1 - s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 &= -f(b, d), \\
 & (b - a)(d - c) \int_0^{\frac{1}{1 + q_1}} \int_0^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(tb + (1 - t)a, sd + (1 - s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 &= -f\left(\frac{q_1 a + b}{1 + q_1}, d\right), \\
 & (b - a)(d - c) \int_0^{\frac{1}{1 + q_1}} \int_0^1 \frac{{}_a c \partial_{q_1, q_2}^2 f(tb + (1 - t)a, sd + (1 - s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 &= -f\left(b, \frac{q_2 c + d}{1 + q_2}\right), \\
 & (b - a)(d - c) \int_0^{\frac{1}{1 + q_1}} \int_0^{\frac{1}{1 + q_2}} \frac{{}_a c \partial_{q_1, q_2}^2 f(tb + (1 - t)a, sd + (1 - s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} {}_0 d_{q_2} s {}_0 d_{q_1} t \\
 &= f\left(\frac{q_1 a + b}{1 + q_1}, \frac{q_2 c + d}{1 + q_2}\right).
 \end{aligned}
 \tag{3.22}$$

A combination of (3.16)-(3.25), we have (3.15). Thus the proof is accomplished. ■

Remark 2. In Lemma 2, if one takes limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then one has the Lemma 1.

Theorem 5. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}_a c \partial_{q_1, q_2}^2 f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s}$ is continuous and integrable on Δ and $\left| \frac{{}_a c \partial_{q_1, q_2}^2 f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r$ is convex on the co-ordinates on Δ for $r \geq 0$, then the following inequality holds:

$$\begin{aligned}
 & |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b - a)(d - c) C_1^{1 - \frac{1}{r}}(q_1, q_2) \\
 & \times \left[\left| \frac{{}_a c \partial_{q_1, q_2}^2 f(b, d)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r C_2(q_1, q_2) + \left| \frac{{}_a c \partial_{q_1, q_2}^2 f(a, d)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r C_3(q_1, q_2) \right. \\
 & \left. + \left| \frac{{}_a c \partial_{q_1, q_2}^2 f(b, c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r C_4(q_1, q_2) + \left| \frac{{}_a c \partial_{q_1, q_2}^2 f(a, c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r C_5(q_1, q_2) \right]^{\frac{1}{r}}
 \end{aligned}
 \tag{3.26}$$

where

$$\begin{aligned}
 C_1(q_1, q_2) &= \frac{4}{(1 + q_1)^3 (1 + q_2)^3}, \\
 C_2(q_1, q_2) &= \frac{(-q_1^3 q_2^3 - q_1^3 q_2^2 + 5q_1^3 q_2 - q_1^2 q_2^3 + 6q_1^2 q_2 + q_1^2)}{q_1 q_2 (1 + q_1)^3 (1 + q_2)^3 (1 + q_1 + q_1^2)(1 + q_2 + q_2^2)}, \\
 C_3(q_1, q_2) &= \frac{(-q_2^3 q_1^3 - q_2^3 q_1^2 + 5q_2^3 q_1 - q_2^2 q_1^3 + 6q_2^2 q_1 + q_2^2)}{q_1 q_2 (1 + q_1)^3 (1 + q_2)^3 (1 + q_1 + q_1^2)(1 + q_2 + q_2^2)}, \\
 C_4(q_1, q_2) &= \frac{(-q_2^3 q_1^3 - 3q_2 q_1 + q_2 + q_1^2 + q_1 + 1)}{q_1 q_2 (1 + q_1)^3 (1 + q_2)^3 (1 + q_1 + q_1^2)(1 + q_2 + q_2^2)},
 \end{aligned}$$

$$C_4(q_1, q_2) = \frac{\begin{pmatrix} -2q_1^5q_2^3 - 2q_1^5q_2 - 2q_1^4q_2^3 - 4q_1^4q_2^2 - 4q_1^4q_2 - 6q_1^4 + 2q_1^3q_2^4 + 16q_1^3q_2^3 \\ + 10q_1^3q_2^2 + 2q_1^3q_2 + 6q_1^3 - 2q_1^2q_2^5 - 4q_1^2q_2^4 + 10q_1^2q_2^3 + 8q_1^2q_2^2 + 4q_1^2q_2 - 4q_1^2 \\ - 2q_1q_2^5 - 4q_1q_2^4 + 2q_1q_2^3 + 4q_1q_2^2 + 9q_1q_2 - 2q_2^5 - 6q_2^4 - 6q_2^3 - 4q_2^2 \end{pmatrix}}{q_1q_2(1+q_1)^3(1+q_2)^3(1+q_1+q_1^2)(1+q_2+q_2^2)}.$$

Proof. Taking the absolute value on both sides of the equality (3.15), using q_1q_2 -power mean inequality for functions of two variables (see [10, Theorem 5]) and the convexity of $\left| \frac{a,c\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r$ (see Definition 1) on the co-ordinates on Δ , we have

$$\begin{aligned} & |\mu_{q_1,q_2}(a,b,c,d)(f)| \leq q_1q_2(b-a)(d-c) \\ & \times \left[\int_0^1 \int_0^1 |\kappa(t,s)| \left| \frac{a,c\partial_{q_1,q_2}^2 f(tb + (1-t)a, sd + (1-s)c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right| {}_0d_{q_2}s \ {}_0d_{q_1}t \right] \\ & \leq q_1q_2(b-a)(d-c) \left(\int_0^1 \int_0^1 |\kappa(t,s)| {}_0d_{q_2}s \ {}_0d_{q_1}t \right)^{1-\frac{1}{r}} \\ & \times \left(\int_0^1 \int_0^1 |\kappa(t,s)| \left| \frac{a,c\partial_{q_1,q_2}^2 f(tb + (1-t)a, sd + (1-s)c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r {}_0d_{q_2}s \ {}_0d_{q_1}t \right)^{\frac{1}{r}} \\ & = q_1q_2(b-a)(d-c) \left(\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} ts {}_0d_{q_2}s \ {}_0d_{q_1}t \right. \\ & \quad + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) {}_0d_{q_2}s \ {}_0d_{q_1}t \\ & \quad + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) {}_0d_{q_2}s \ {}_0d_{q_1}t \\ & \quad \left. + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) {}_0d_{q_2}s \ {}_0d_{q_1}t \right) \\ & \times \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} ts \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r {}_0d_{q_2}s \ {}_0d_{q_1}t \right. \\ & \quad + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 s(1-t) \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r {}_0d_{q_2}s \ {}_0d_{q_1}t \\ & \quad + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} t(1-s) \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r {}_0d_{q_2}s \ {}_0d_{q_1}t \\ & \quad \left. + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 (1-t)(1-s) \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r {}_0d_{q_2}s \ {}_0d_{q_1}t \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) \left[\begin{array}{c} ts \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ s(1-t) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ t(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ (1-t)(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \end{array} \right] {}_0 d_{q_2} s \ {}_0 d_{q_1} t \\
 & + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) \left[\begin{array}{c} ts \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ s(1-t) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ t(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ (1-t)(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \end{array} \right] {}_0 d_{q_2} s \ {}_0 d_{q_1} t \\
 & + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) \left[\begin{array}{c} ts \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ s(1-t) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ t(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \\ (1-t)(1-s) \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r \end{array} \right] {}_0 d_{q_2} s \ {}_0 d_{q_1} t \Bigg]^{\frac{1}{r}} \\
 & = q_1 q_2 (b-a)(d-c) C_1^{1-\frac{1}{r}}(q_1, q_2) \\
 & \times \left[\left| \frac{a,c \partial_{q_1,q_2}^2 f(b,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r C_2(q_1, q_2) + \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,d)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r C_3(q_1, q_2) \right. \\
 & \left. + \left| \frac{a,c \partial_{q_1,q_2}^2 f(b,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r C_4(q_1, q_2) + \left| \frac{a,c \partial_{q_1,q_2}^2 f(a,c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r C_5(q_1, q_2) \right]^{\frac{1}{r}}.
 \end{aligned}$$

Note that, in order to compute the coefficients $C_1(q_1, q_2)$, $C_2(q_1, q_2)$, $C_3(q_1, q_2)$, $C_4(q_1, q_2)$ and $C_5(q_1, q_2)$, we calculate 16, 16, 25, 25 and 36 double quantum integrals respectively. In terms of brevity, we omit the details. Thus the proof is accomplished. ■

Corollary 1. In Theorem 5,

(1) If one takes limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then one has

$$\begin{aligned}
 & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
 & \left. - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \, dy - \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \, dx \right| \leq \frac{(b-a)(d-c)}{16} \\
 & \times \left[\frac{\left| \frac{\partial^2 f(b,d)}{\partial t \partial s} \right|^r + \left| \frac{\partial^2 f(a,d)}{\partial t \partial s} \right|^r + \left| \frac{\partial^2 f(b,c)}{\partial t \partial s} \right|^r + \left| \frac{\partial^2 f(a,c)}{\partial t \partial s} \right|^r}{4} \right]^{\frac{1}{r}},
 \end{aligned} \tag{3.27}$$

(2) If one takes $r = 1$, limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then one has

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy - \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right| \leq \frac{(b-a)(d-c)}{16} \times \left[\frac{\left| \frac{\partial^2 f(b,d)}{\partial t \partial s} \right| + \left| \frac{\partial^2 f(a,d)}{\partial t \partial s} \right| + \left| \frac{\partial^2 f(b,c)}{\partial t \partial s} \right| + \left| \frac{\partial^2 f(a,c)}{\partial t \partial s} \right|}{4} \right]^{\frac{1}{r}} \quad (3.28)$$

Remark 3. In (3.27) we recapture [8, Theorem 4, inequality (2.13)], in (3.28) we recapture [8, Theorem 4, inequality (2.4)].

Theorem 6. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{a,c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t c \partial_{q_2} s}$ is continuous and integrable on Δ and $\left| \frac{a,c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r$ is convex on the co-ordinates on Δ for $r > 0$, then the following inequality holds:

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b-a)(d-c) \left(\int_0^1 \int_0^1 |\kappa(t, s)|^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right)^{\frac{1}{p}} \times \left[\frac{\left| \frac{a,c \partial_{q_1, q_2}^2 f(b,d)}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r + q_1 \left| \frac{a,c \partial_{q_1, q_2}^2 f(a,d)}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r + q_2 \left| \frac{a,c \partial_{q_1, q_2}^2 f(b,c)}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r + q_1 q_2 \left| \frac{a,c \partial_{q_1, q_2}^2 f(a,c)}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r}{(1+q_1)(1+q_2)} \right]^{\frac{1}{r}}. \quad (3.29)$$

where $\kappa(t, s)$ is the same in Lemma 2 and $\frac{1}{p} + \frac{1}{r} = 1$.

Proof. Taking the absolute value on both sides of the equality (3.15), using $q_1 q_2$ -Hölder inequality for functions of two variables (see [10, Theorem 5]) and the convexity of $\left| \frac{a,c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r$ (see Definition 1) on the co-ordinates on Δ , we have

$$\begin{aligned} |\mu_{q_1, q_2}(a, b, c, d)(f)| &\leq q_1 q_2 (b-a)(d-c) \\ &\times \left[\int_0^1 \int_0^1 |\kappa(t, s)| \left| \frac{a,c \partial_{q_1, q_2}^2 f(tb + (1-t)a, sd + (1-s)c)}{a \partial_{q_1} t c \partial_{q_2} s} \right| {}_0 d_{q_2} s {}_0 d_{q_1} t \right] \\ &\leq q_1 q_2 (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\kappa(t, s)|^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right)^{\frac{1}{p}} \right. \\ &\times \left. \left(\int_0^1 \int_0^1 |\kappa(t, s)| \left| \frac{a,c \partial_{q_1, q_2}^2 f(tb + (1-t)a, sd + (1-s)c)}{a \partial_{q_1} t c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s {}_0 d_{q_1} t \right)^{\frac{1}{r}} \right] \\ &\leq q_1 q_2 (b-a)(d-c) \left[\left(\int_0^1 \int_0^1 |\kappa(t, s)|^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right)^{\frac{1}{p}} \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\left| \frac{a,c\partial_{q_1,q_2}^2 f(b,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r \int_0^1 \int_0^1 ts {}_0d_{q_2}s {}_0d_{q_1}t \right. \\
& + \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r \int_0^1 \int_0^1 s(1-t) {}_0d_{q_2}s {}_0d_{q_1}t \\
& + \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r \int_0^1 \int_0^1 t(1-s) {}_0d_{q_2}s {}_0d_{q_1}t \\
& \left. + \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r \int_0^1 \int_0^1 (1-t)(1-s) {}_0d_{q_2}s {}_0d_{q_1}t \right]^{\frac{1}{r}} \\
& = q_1q_2(b-a)(d-c) \left(\int_0^1 \int_0^1 |\kappa(t,s)|^p {}_0d_{q_2}s {}_0d_{q_1}t \right)^{\frac{1}{p}} \\
& \times \left[\frac{\left| \frac{a,c\partial_{q_1,q_2}^2 f(b,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r + q_1 \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r + q_2 \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r + q_1q_2 \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r}{(1+q_1)(1+q_2)} \right]^{\frac{1}{r}}.
\end{aligned}$$

Thus the proof is accomplished. \blacksquare

Remark 4. In theorem 6, if one takes limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then one has [8, Theorem 3].

In terms of brevity, we will use the following notations

$$\begin{aligned}
L &= \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \quad M = \left| \frac{a,c\partial_{q_1,q_2}^2 f(a,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \quad N = \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \quad O = \left| \frac{a,c\partial_{q_1,q_2}^2 f(b,d)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \\
P &= \left| \frac{a,c\partial_{q_1,q_2}^2 f\left(a, \frac{q_2c+d}{1+q_2}\right)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \quad Q = \left| \frac{a,c\partial_{q_1,q_2}^2 f\left(b, \frac{q_2c+d}{1+q_2}\right)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \quad R = \left| \frac{a,c\partial_{q_1,q_2}^2 f\left(\frac{q_1a+b}{1+q_1}, c\right)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|, \\
S &= \left| \frac{a,c\partial_{q_1,q_2}^2 f\left(\frac{q_1a+b}{1+q_1}, d\right)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right| \quad \text{and} \quad T = \left| \frac{a,c\partial_{q_1,q_2}^2 f\left(\frac{q_1a+b}{1+q_1}, \frac{q_2c+d}{1+q_2}\right)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|.
\end{aligned}$$

Theorem 7. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partially q_1q_2 -differentiable function on Δ° . If partial q_1q_2 -derivative $\frac{a,c\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t \ c\partial_{q_2}s}$ is continuous and integrable on Δ and $\left| \frac{a,c\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r$ is quasi-convex on the co-ordinates on Δ for $r \geq 0$, then the following inequality holds:

$$|\mu_{q_1,q_2}(a,b,c,d)(f)| \leq (b-a)(d-c) \frac{q_1q_2}{(1+q_1)^3(1+q_2)^3} (A+B+C+D) \quad (3.30)$$

where $A = \sup\{L, P, R, T\}$, $B = \sup\{N, R, Q, T\}$, $C = \sup\{M, P, S, T\}$ and $D = \sup\{O, Q, S, T\}$.

Proof. Taking the absolute value on both sides of the equality (3.15), using q_1q_2 -power mean inequality for functions of two variables (see [10, Theorem 5]) and the quasi-convexity of

$\left| \frac{a,c\partial_{q_1,q_2}^2 f}{a\partial_{q_1}t \ c\partial_{q_2}s} \right|^r$ (see Definition 1) on the co-ordinates on Δ , we have

$$\begin{aligned}
& |\mu_{q_1,q_2}(a,b,c,d)(f)| \leq q_1q_2(b-a)(d-c) \\
& \times \left[\int_0^1 \int_0^1 |\kappa(t,s)| \left| \frac{a,c\partial_{q_1,q_2}^2 f(tb + (1-t)a, sd + (1-s)c)}{a\partial_{q_1}t \ c\partial_{q_2}s} \right| {}_0d_{q_2}s {}_0d_{q_1}t \right] \\
& = q_1q_2(b-a)(d-c)
\end{aligned}$$

$$\begin{aligned}
 & \times \left[\begin{aligned} & \int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t s \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right| {}_0 d_{q_2} s \, {}_0 d_{q_1} t \\ & + \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right| {}_0 d_{q_2} s \, {}_0 d_{q_1} t \\ & + \int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right| {}_0 d_{q_2} s \, {}_0 d_{q_1} t \\ & + \int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right| {}_0 d_{q_2} s \, {}_0 d_{q_1} t \end{aligned} \right] \\
 & \leq q_1 q_2 (b-a)(d-c) \\
 & \times \left(\left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t s {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{1-\frac{1}{r}} \right. \\
 & \left. \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t s \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{1-\frac{1}{r}} \right. \\
 & \left. \left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{1-\frac{1}{r}} \right. \\
 & \left. \left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{1-\frac{1}{r}} \right. \\
 & \left. \left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) \left| \frac{a_c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \, c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & \leq q_1 q_2 (b-a)(d-c) \left[\sup\{L, P, R, T\} \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t s {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right] \right. \\
 & \quad + \sup\{N, R, Q, T\} \left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s \left(\frac{1}{q_1} - t \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right] \\
 & \quad + \sup\{M, P, S, T\} \left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right] \\
 & \quad \left. + \sup\{O, Q, S, T\} \left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) {}_0 d_{q_2} s \, {}_0 d_{q_1} t \right] \right] \\
 & = (b-a)(d-c) \frac{q_1 q_2}{(1+q_1)^3 (1+q_2)^3} (A + B + C + D).
 \end{aligned}$$

Thus the proof is accomplished. ■

Corollary 2. Suppose the conditions of the Theorem 7 are satisfied. Additionally if

$$\begin{aligned}
 (1) \quad & \left| \frac{a_c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t \, c \partial_{q_2} s} \right| \text{ is increasing on the co-ordinates on } \Delta, \text{ then} \\
 & |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq (b-a)(d-c) \frac{q_1 q_2}{(1+q_1)^3 (1+q_2)^3} (O + Q + S + T), \quad (3.31)
 \end{aligned}$$

$$(2) \left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right| \text{ is decreasing on the co-ordinates on } \Delta, \text{ then}$$

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq (b-a)(d-c) \frac{q_1 q_2}{(1+q_1)^3 (1+q_2)^3} (L+R+P+T), \quad (3.32)$$

$$(3) T=R=S=P=Q=0, \text{ then}$$

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq (b-a)(d-c) \frac{q_1 q_2}{(1+q_1)^3 (1+q_2)^3} (L+M+N+O), \quad (3.33)$$

$$(4) L=M=N=O=P=Q=R=S=0, \text{ then}$$

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq (b-a)(d-c) \frac{q_1 q_2}{(1+q_1)^3 (1+q_2)^3} T. \quad (3.34)$$

Remark 5. In Theorem 7, if we take limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then we recapture [9, Theorem 2 and Theorem 4], in Corollary 2, if we take limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then we recapture [9, Corollary 1 and Corollary 3].

Theorem 8. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partially $q_1 q_2$ -differentiable function on Δ° . If partial $q_1 q_2$ -derivative $\frac{{}_a c \partial_{q_1}^2 {}_{q_2} f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s}$ is continuous and integrable on Δ and $\left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r$ is quasi-convex on the co-ordinates on Δ for $r > 0$, then the following inequality holds:

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b-a)(d-c) \quad (3.35)$$

$$\times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} A + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} B \right]$$

$$+ C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} C + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} D$$

where A, B, C, D are the same in Theorem 7

$$C_6(q_1, q_2, p) = \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t^p s^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right]^{\frac{1}{p}},$$

$$C_7(q_1, q_2, p) = \left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s^p \left(\frac{1}{q_1} - t \right)^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right]^{\frac{1}{p}},$$

$$C_8(q_1, q_2, p) = \left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t^p \left(\frac{1}{q_2} - s \right)^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right]^{\frac{1}{p}},$$

$$C_9(q_1, q_2, p) = \left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right)^p \left(\frac{1}{q_2} - s \right)^p {}_0 d_{q_2} s {}_0 d_{q_1} t \right]^{\frac{1}{p}}.$$

$$\text{and } \frac{1}{p} + \frac{1}{r} = 1.$$

Proof. Taking the absolute value on both sides of the equality (3.15), using $q_1 q_2$ -Hölder inequality for functions of two variables (see [10, Theorem 5]) and the quasi-convexity of $\left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right|^r$ (see Definition 1) on the co-ordinates on Δ , we have

$$|\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b-a)(d-c)$$

$$\times \left[\int_0^1 \int_0^1 |\kappa(t, s)| \left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right| {}_0 d_{q_2} s {}_0 d_{q_1} t \right]$$

$$= q_1 q_2 (b-a)(d-c)$$

$$\times \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t s \left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right| {}_0 d_{q_2} s {}_0 d_{q_1} t \right]$$

$$+ \int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t \left(\frac{1}{q_2} - s \right) \left| \frac{{}_a c \partial_{q_1}^2 {}_{q_2} f(t b + (1-t)a, s d + (1-s)c)}{{}_a \partial_{q_1} t {}_c \partial_{q_2} s} \right| {}_0 d_{q_2} s {}_0 d_{q_1} t$$

$$\begin{aligned}
 & + \frac{1}{1+q_1} \int_0^{1+q_2} s \left(\frac{1}{q_1} - t \right) \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right| {}_0 d_{q_2} s \ {}_0 d_{q_1} t \\
 & + \frac{1}{1+q_1} \int_{\frac{1}{1+q_1}}^{\frac{1}{1+q_2}} \left(\frac{1}{q_1} - t \right) \left(\frac{1}{q_2} - s \right) \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right| {}_0 d_{q_2} s \ {}_0 d_{q_1} t \Big] \\
 & \leq q_1 q_2 (b-a)(d-c) \\
 & \times \left[\left(\left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} t^p s^p {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{p}} \right. \right. \\
 & \left. \left[\int_0^{\frac{1}{1+q_1}} \int_0^{\frac{1}{1+q_2}} \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 t^p \left(\frac{1}{q_2} - s \right)^p {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{p}} \right. \\
 & \left. \left[\int_0^{\frac{1}{1+q_1}} \int_{\frac{1}{1+q_2}}^1 \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} s^p \left(\frac{1}{q_1} - t \right)^p {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{p}} \right. \\
 & \left. \left[\int_{\frac{1}{1+q_1}}^1 \int_0^{\frac{1}{1+q_2}} \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \\
 & + \left(\left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left(\frac{1}{q_1} - t \right)^p \left(\frac{1}{q_2} - s \right)^p {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{p}} \right. \\
 & \left. \left[\int_{\frac{1}{1+q_1}}^1 \int_{\frac{1}{1+q_2}}^1 \left| \frac{a, c \partial_{q_1, q_2}^2 f(t b + (1-t)a, s d + (1-s)c)}{a \partial_{q_1} t \ c \partial_{q_2} s} \right|^r {}_0 d_{q_2} s \ {}_0 d_{q_1} t \right]^{\frac{1}{r}} \right) \Big] \\
 & \leq q_1 q_2 (b-a)(d-c) \\
 & \times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} A + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} B \right. \\
 & \left. + C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} C + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} D \right].
 \end{aligned}$$

Thus the proof is accomplished. ■

Corollary 3. Suppose the conditions of the Theorem 8 are satisfied. Additionally if

$$\begin{aligned}
 (1) \quad & \left| \frac{a, c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t \ c \partial_{q_2} s} \right| \text{ is increasing on the co-ordinates on } \Delta, \text{ then} \\
 & |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b-a)(d-c) \\
 & \times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} O + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} Q \right. \\
 & \left. + C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} S + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} T \right],
 \end{aligned} \tag{3.36}$$

$$\begin{aligned}
 (2) \quad & \left| \frac{a, c \partial_{q_1, q_2}^2 f}{a \partial_{q_1} t \ c \partial_{q_2} s} \right| \text{ is decreasing on the co-ordinates on } \Delta, \text{ then} \\
 & |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b-a)(d-c)
 \end{aligned} \tag{3.37}$$

$$\times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} L + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} R \right. \\ \left. + C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} P + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} T \right] \\ (3) \ T = R = S = P = Q = 0, \text{ then} \\ |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b - a)(d - c) \quad (3.38)$$

$$\times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} L + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} M \right. \\ \left. + C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} N + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} O \right] \\ (4) \ L = M = N = O = P = Q = R = S = 0, \text{ then} \\ |\mu_{q_1, q_2}(a, b, c, d)(f)| \leq q_1 q_2 (b - a)(d - c)T \quad (3.39)$$

$$\times \left[C_6(q_1, q_2, p) \left(\frac{1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} + C_7(q_1, q_2, p) \left(\frac{q_1}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} \right. \\ \left. + C_8(q_1, q_2, p) \left(\frac{q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} + C_9(q_1, q_2, p) \left(\frac{q_1 q_2}{(1+q_1)(1+q_2)} \right)^{\frac{1}{r}} \right]$$

Remark 6. In Theorem 8, if we take limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then we recapture [9, Theorem 3], in Corollary 3, if we take limit $q_1^- \rightarrow 1$ and $q_2^- \rightarrow 1$, then we recapture [9, Corollary 2].

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